Assuming a solution of the form

$$R(\varrho) = \varrho^l [a_0 + a_1 \varrho + a_2 \varrho^2 \dots] e^{-\varkappa \varrho} \tag{4}$$

results, by ordering in power of ϱ, in

$$\sum_{\nu=0} \varrho^\nu [(\nu+1)(\nu+2l+2) a_{\nu+1} + (\lambda - 2\varkappa(l+1) - 2\varkappa\nu) a_\nu] = 0 . \tag{5}$$

The usual termination condition on the polynomial $\Sigma a_\nu \varrho^\nu$ at a finite ν, say $\nu = n_r$, gives the condition for the alowed values λ,

$$\lambda = 2\varkappa(n_r + l + 1) = N^* \tag{6}$$

where the choice of $\varkappa$ is still arbitrary. Any $\varkappa > 0$ ensures the quadratic integrability of $R(\varrho)$.

Equation (2b) suggests, that k may be identified with a wave number which by Eq. (2 a) is confined to the values

$$k_{n^*} = \varkappa k_0/\lambda = k_0/n^*, \quad k_0 = 2\mu Z e^2/\hbar \tag{7}$$

which are with

$$\lambda/\varkappa = n^* = 2(n_r + l + 1) = 2, 4, 6, \dots \tag{8}$$

independent of $\varkappa$. From Eqs. (7) and (2 b) the allowed energies

$$E_{n^*} = -|E_{n^*}| = -E_0/n^{*2}, \quad E_0 = 2\mu Z^2 e^4/\hbar^2 \tag{9}$$

are thus independent of $\varkappa$ also.

The sequence of energies E_{n^*}

$$E_2 = -E_0/4, \quad E_4 = -E_0/16, \quad E_6 = -E_0/36, \dots \tag{10}$$

is a universal property of the Coulomb potential and is, therefore, apart from the different labelling, in particular identical to that of the H-atom. In the conventional treatment of the H-atom, however, one introduces a principal quantum number n,

$$n = n^*/2 \tag{11}$$

and then labels the energies by n, $E_n = -E_0/4n^2$ with $n = 1, 2, 3, \dots$. It appears that n has to be considered a short notation for $n^*/2$.

Since the energies are independent on $\varkappa$, they cannot be used for distinguishing spectra with respect to different $\varkappa$. However, because of Eq. (6) a suitable classification is possible in terms of the quantum number $\lambda(\varkappa)$.

4.2. Classes of Spectra

Let us consider the two cases $\varkappa = 1/2$ and $\varkappa = 1$. The first case corresponds to the generation of the hydrogen spectrum, the second will turn out to correspond to the generation of the PSE-spectrum.

From Eq. (6) we have for $\varkappa = 1/2$:

$$\lambda = (n_r + l + 1) = N^*, \quad N^* = 1, 2, 3, \dots . \tag{12}$$

By Eqs. (8) and (11) holds, rather incidentically, $N^* = n$. Therefore, the conventional classification of the hydrogen states in terms of n conceals the significance of $\varkappa$ in the generation of the spectrum.

In case of the PSE however, the rôle of $\varkappa$ seems to be of some importance. From Eq. (6) we have for $\varkappa = 1$:

$$\lambda = 2(\bar{n}_r + l + 1) = N^*, \quad N^* = 2, 4, 6, \dots \tag{13a}$$

where the bar-notation is used simply for the purpose of differentiation from the case $\varkappa = 1/2$. Since l and $\bar{n}_r$ are by Eq. (4) integer numbers, N^* can indeed assume only even numbers.

Each of the states with $N^* = \text{const}$ carries a contribution of twice the angular momentum quantum number l and this is exactly the property which is characteristic for the PSE. However, in the conventional classification of Fig. 2 the principal quantum number $n = n_r + l + 1$ is used and Eq. (13 a) reads

$$\lambda = n + l + 1 = n_r + 2l + 2 = N + 1. \tag{13b}$$

By comparison of Eqs. (13 a) and (13 b) one obtains

$$n_r = 2\bar{n}_r, \quad N + 1 = N^*. \tag{14, 15}$$

The relationship (14) has to be considered as the definition of a short notation for $2\bar{n}_r$ and hence n_r must be even. Similarly, by Eq. (15), $N+1$ can assume only even values corresponding to the allowed values of N^* given in Eq. (13 a). Apparently, the classification of the PSE given in Fig. 2 contradicts in part these conditions. In fact, only every second shell of Fig. 2 survives if these conditions are imposed and one obtains the spectrum given in Fig. 5 a which may be written in terms of $\bar{n}_r$ and l as given in Fig. 5 b.

4 - SHELL-CLASSIFICATIONS

n, ℓ				$\tilde{N}^*$	$\bar{n}_r, \ell$				$\tilde{N}$	$\bar{n}, \ell$			
4f	5d	6p	7s	8	0,3	1,2	2,1	3,0	3	4f	4d	4p	4s
	3d	4p	5s	6		0,2	1,1	2,0	2		3d	3p	3s
		2p	3s	4			0,1	1,0	1			2p	2s
			1s	2				0,0	0				1s

Fig. 5. 4-Shell-Classification of the PSE in terms of (from left to right) a) the conventional quantum numbers n and $\underline{l}$; b) the PSE-quantum numbers $\overline{n_r}$ and $\underline{l}$; $\underline{c)}$ the PSE-quantum-numbers $\overline{n} = \overline{n_r} + l + 1$ and l.